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Long-term optimal portfolio allocation under dynamic horizon-specific risk aversion

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Abstract

This paper studies the long-term asset allocation problem of an individual with risk aversion coefficient that i) varies with economic conditions, and ii) exhibits different risk attitudes towards the short and the long term. To do this, we propose a parametric linear portfolio policy that accommodates an arbitrarily large number of assets in the portfolio and a piecewise linear risk aversion coefficient. These specifications of the optimal portfolio policy and individual's risk aversion allow us to apply GMM methods for parameter estimation and testing. Our empirical results provide statistical evidence of the existence of a short-term and a long-term regime in the individual's risk aversion. Long-term risk aversion is always higher than short-term risk aversion, and it is more statistically significant as the investment horizon increases. The analysis of the optimal portfolio weights also suggests that the allocation to stocks and bonds is strongly negatively correlated, with the magnitude of the portfolio weights and risk aversion coefficients increasing as the investment horizon expands.

Keywords: dynamic risk aversion; intertemporal portfolio theory; parametric portfolio policies; threshold nonlinearity tests.

JEL Codes: E32, E52, E62.

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1 Introduction

Optimal portfolio decisions depend on the details of the economic and financial environment: the financial assets that are available, their expected returns and risks, and the preferences and circumstances of investors. These details become particularly relevant for long-term investors. Such investors must concern themselves not only with expected returns and risks today, but with the way in which expected returns and risks may change over time. It is widely understood at least since the work of Samuelson (1969) and Merton (1969, 1971, 1973) that the solution to a multiperiod portfolio choice problem can be very different from the solution to a static portfolio choice problem. Unfortunately, intertemporal asset allocation models are hard to solve in closed form unless strong assumptions on the investor's objective function such as log preferences or a lognormal distribution for asset returns are imposed. This situation has begun to change as a result of several developments in numerical methods and continuous time finance models such as Barberis (2000) and Brennan et al. (1997, 1999), amongst other authors. Approximate analytical solutions to the Merton model have been developed in Campbell and Viceira (1999, 2001, 2002) and Campbell et al. (2003) for models exhibiting an intertemporal elasticity of substitution close to one. Recent parametric alternatives to solving the investor optimal portfolio problem have been proposed by Brandt (1999), Aït-Sahalia and Brandt (2001) and Brandt and Santa-Clara (2006).

These seminal contributions on optimal asset allocation for long-term investors assume in most cases an exogenous and constant risk aversion coefficient to model individuals' risk attitude. This assumption can be too simplistic in frameworks characterized by economic uncertainty and rapidly changing investment environments. It may be more realistic to consider investor's preferences to be dynamic and influenced by the economic landscape. More specifically, a more uncertain economic environment can lead individuals to consider more cautiously the same investment opportunities than under a favourable environment, hence, it is plausible

to interpret the investment decisions of these individuals as strongly shaped by their views on future economic conditions and their attitudes towards short-term and long-term uncertainty. Recent examples of the differences in attitude towards the short and the long term can be found in individuals' views of potential outcomes of political and socioeconomic decisions, e.g. political elections, Brexit, or, more specifically, financial crises. These differences in risk aversion between the short and the long term have been, however, largely unexplored in the long-term optimal portfolio literature. These differences have just been captured by distinguishing between single-period and multiperiod utility functions, however, no allowance has been made to the possibility of long-term investors with differing views between the short and the long term.

The possibility of dynamics in the relative risk aversion coefficient has not received much attention either in the investment and asset pricing literature. Notable exceptions are Campbell and Cochrane (1999), Chan and Kogan (2002), and more recently, Brandt and Wang (2003). In these seminal contributions, the representative agent's relative risk aversion coefficient varies with the difference between consumption and the agent's habit. This habit can be interpreted as a minimum subsistence level required by the individual or some dynamic value that is formed through past consumption.

The aim of this paper is to extend the above literature on multiperiod asset allocation to accommodate dynamics in risk aversion and the possibility of different perceptions towards the short and the long term. To do this, we develop an analytical framework that allows us to estimate the optimal portfolio weights of investors with multiperiod investment horizons. In contrast to the related literature, the risk aversion coefficient is modelled as a piecewise linear function defined over the individual's investment horizon and separating the short from the long term. Both regimes accommodate the presence of dynamics in risk aversion that are driven by economic conditions. Intuitively, the individual exhibits different risk attitudes to negative events taking place before the structural break period separating the short from

the long term. The choice of a piecewise linear function for modeling the dynamics of risk aversion is formalized by proposing a likelihood ratio test comparing the suitability of the linear and nonlinear risk aversion specifications. The econometric methodology is similar in spirit to the seminal papers by Davies (1977, 1987), Andrews (1993), Andrews and Ploberger (1994) and Hansen (1996) that discuss how to make inference when a nuisance parameter is not identified under the null hypothesis. More specifically, in our setting, we assume the period separating the short term from the long term to be unknown, and is estimated from the data. Under the null hypothesis, there is a single risk aversion regime implying that the period signalling the structural break (nuisance parameter) is not identified. In this scenario, standard statistical inference procedures cannot be applied to statistically assess the presence of a threshold nonlinearity. Instead, we apply a p-value transformation method implemented through a multiplier method to the first order conditions of the individual's maximization problem, see Hansen (1996) for early applications of the methodology and, more recently, Chernozhukov et al. (2016) for nonparametric bootstrap versions of the method.

The existence of a multiperiod optimal decision problem implies in most cases the lack of closed form solutions and the need of dynamic stochastic programming methods. To overcome this problem, we take advantage of the methodology proposed in Aït-Sahalia and Brandt (2001) and assume that the optimal portfolio weights are driven by a parametric linear policy rule. This technique allows us to obtain an overidentified set of first order conditions of the multiperiod optimization problem that is exploited for parameter estimation and hypothesis testing. More specifically, we can estimate the marginal contributions of the state variables to the risk aversion function and the dynamic optimal portfolio weights using the sample counterparts of the multiple Euler equations that characterize the optimal portfolio choice. The correct specification of the parametric policy rules can be tested using a version of the overidentification J-test developed by Hansen (1982). Estimation involves two different sets of parameters: a vector that

characterises the optimal portfolio choice and a vector that characterises the dynamics of the risk aversion coefficient. Both sets are simultaneously estimated using an extended version of the generalized method of moments (GMM) procedure developed in this paper.

This methodology is explored in an empirical application assessing the optimal portfolio decisions of a long-term investor with different investment horizons holding a tactical portfolio given by stocks, bonds and cash spanning thirty years of financial returns. This empirical exercise closely follows similar studies such as Brennan et al. (1997), Brandt (1999) and Campbell et al. (2003), among many others. The investor is assumed to invest in three assets - a one-month Treasury bill as riskless security, a long-term bond (G0Q0 index), and an equity portfolio (S&P 500 index). We consider a set of state variables that is common in the predictive literature on asset pricing and portfolio theory: the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of excess stock and bond returns.

Our empirical results provide statistical evidence of the existence of a short-term and a long-term regime in individual's risk aversion. Our findings suggest that the period differentiating the short from the long term is the seven month of the investment horizon characterizing the individual's multiperiod objective function. These findings also reveal that long-term risk aversion is higher than short-term risk aversion, and increases with the number of periods defining the individual's investment horizon. The analysis of the optimal portfolio weights also suggests that the allocation to stocks and bonds is strongly negatively correlated, with both allocations driven by a set of state variables reflecting the economic environment. The increase in risk aversion observed during distress episodes highlights flight to quality behaviors from stocks to bonds for investors exhibiting dynamic risk aversion. In these periods, these investors increase significantly their allocation to bonds compared to the allocation of investors exhibiting constant risk aversion. As a byproduct of our analysis, we find a positive relationship between the magnitude of the portfolio weights and the investment horizon. This finding suggests that

investors consider more aggressive short and long positions on the portfolio as the investment horizon is further into the future and covers more investment periods. We also observe a positive relationship between the degree of investor's risk aversion and the investment horizon suggesting that individuals with investment plans constructed over longer periods are more risk averse than myopic investors facing the same investment opportunities.

The rest of the article is structured as follows. Section 2 presents the model and derives the system of overidentified equations corresponding to the first order conditions of the multiperiod maximization problem of an individual exhibiting a time-varying piecewise linear risk aversion coefficient. Section 3 discusses the implementation of GMM to estimate the optimal portfolio weights and the risk aversion coefficients and briefly discusses the corresponding asymptotic theory. Section 4 presents two types of econometric tests to assess the parametric assumptions used in the development of our model. First, we introduce in detail a threshold nonlinearity test to assess statistically the existence of piecewise nonlinearities in the individual's strategic multiperiod utility function, and second, we discuss several specification tests to assess the suitability of the parametric policy rules proposed in the paper. Section 5 presents an empirical application to derive the optimal allocation to a portfolio of stocks, bonds and cash for a strategic investor with a multiperiod utility function. Section 6 concludes.

2 The Model

2.1 The investor's multiperiod objective function

Consider the portfolio choice of an investor who maximizes the expected utility of real wealth (w_t) over K multiple periods. Assume that the utility function is additively time separable and

takes the form

$$\sum_{j=0}^K \beta^j E_t \left[\frac{w_{t+j}^{1-\gamma(j)}}{1-\gamma(j)} \right], \quad (1)$$

with

$$\ln \gamma(j) \equiv \ln \gamma(j, z_{t+j}) = \gamma' z_{t+j} + \eta' z_{t+j} 1(j > k_0) \quad (2)$$

where z_{t+j} is a $n \times 1$ vector comprising a constant and a set of $n - 1$ macroeconomic and financial variables reflecting all the information available to the investor at time $t + j$. This piecewise linear formulation follows the spirit of Gonzalo and Pitarakis (2012, 2016) on threshold predictive regression and Perron (1989, 1997) and Andrews (1993) on structural breaks.

Two parameters describe individuals' preferences: the discount factor β measures patience, the willingness to give up consumption today for consumption tomorrow, and the coefficient $\gamma(j)$ captures risk aversion, the reluctance to trade consumption for a fair gamble over consumption today. The parameters $\gamma = (\gamma_c, \gamma_1, \dots, \gamma_{n-1})'$ and $\eta = (\eta_c, \eta_1, \dots, \eta_{n-1})'$ capture the effect of these variables on the risk aversion coefficient. The dynamics of the risk aversion function are driven by changes in macroeconomic conditions and the individual's risk attitude with respect to the strategic investment horizon. The parameter k_0 denotes the period separating the short from the long term and is defined over the K -period individual's investment horizon. The vector η captures the differences in the risk aversion coefficient between the short and long term. The structural model (2) for the risk aversion coefficient $\gamma(j)$ can be interpreted as an alternative to the stochastic mean-reverting autoregressive process proposed in Brandt and Wang (2003) for modeling relative risk aversion. The above function (2) can be alternatively expressed as

$$\ln \gamma(j) = \gamma' z_{t+j} + \eta' z_{t+j} 1(\omega_j > \omega_0) \quad (3)$$

with $\omega_0, \omega_j \in [\omega_{\min}, \omega_{\max}] \in (0, 1)$ and such that $k_0 = \lceil \omega_0 K \rceil$ where $\lceil \cdot \rceil$ denotes the integer part of the value inside the brackets.

The individual begins life with an exogenous endowment of wealth $w_0 \geq 0$. This endowment accumulates over time according to the equation

$$w_{t+1} = (1 + r_{t+1}^p)w_t. \quad (4)$$

At the beginning of the period $t + 1$ the individual receives income from allocating resources in an investment portfolio offering a real return r_{t+1}^p . The portfolio return is defined as

$$r_{t+1}^p(\alpha_t) = r_{f,t+1} + \alpha_t' r_{t+1}^e, \quad (5)$$

with $r_{t+1}^e = (r_{1,t+1} - r_{f,t+1}, \dots, r_{m,t+1} - r_{f,t+1})'$ denoting the vector of excess returns on the m risky assets over the real risk-free rate $r_{f,t+1}$, and $\alpha_t = (\alpha_{1,t}, \dots, \alpha_{m,t})'$ denoting the different allocations to risky assets. In order to be able to solve a multiperiod maximization problem that accommodates in a parsimonious way arbitrarily long investment horizons, we entertain the parametric portfolio policy rule introduced in the seminal contributions of Aït-Sahalia and Brandt (2001), Brandt and Santa-Clara (2006) and Brandt et al. (2009):

$$\alpha_{h,t+i} = \lambda_h' z_{t+i}, \quad h = 1, \dots, m, \quad (6)$$

with $\lambda_h = (\lambda_{h,1}, \dots, \lambda_{h,n})'$ the vector of parameters associated to the state variables z_t . Time variation of the optimal asset allocation is introduced through the dynamics of the state variables. This specification of the portfolio weights has two main features. First, it allows us to study the marginal effects of the state variables on the optimal portfolio weights through the set of parameters λ , and second, it avoids the introduction of time consuming stochastic dynamic programming methods¹.

¹This approach forces the individual's optimal portfolio policy rule to be linear and with the same parameter values over the long term horizon. More sophisticated models can be developed that entertain different para-

2.2 Optimal portfolio choice under risk aversion

In this section we derive the first order conditions of the long-term optimal portfolio choice problem for a risk-averse individual with preferences described above. The investor's wealth process at time $t + j$ can be expressed in terms of the compound j -period gross return and the initial wealth w_t . More formally,

$$w_{t+j} = \prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) w_t. \quad (7)$$

Using this characterization of the wealth process simple algebra shows that the individual's maximization problem can be written as

$$\max_{\{\lambda_{hs}\}} \left\{ \sum_{j=0}^K E_t \left[\beta^j \frac{w_t^{1-\gamma(j)}}{1-\gamma(j)} \left(\prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) \right)^{1-\gamma(j)} \right] \right\}. \quad (8)$$

The first order conditions of this optimization problem with respect to the vector of parameters λ_{hs} , with $h = 1, \dots, m$ and $s = 1, \dots, n$, provide for each $\omega \in [\omega_{\min}, \omega_{\max}]$ a system of mn equations characterized by the following conditions:

$$E_t \left[\sum_{j=1}^K \beta^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \right] = 0 \quad (9)$$

with

$$\psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) = \left(\sum_{i=1}^j \frac{z_{s,t+i-1} r_{h,t+i}^e}{1 + r_{t+i}^p(\lambda'_h z_{t+i-1})} \right) \left(\prod_{i=1}^j (1 + r_{t+i}^p(\lambda'_h z_{t+i-1})) \right)^{1-\gamma(j)}. \quad (10)$$

metric portfolio policy rules for different investment horizons $i = 1, \dots, K$, however, this approach significantly increases the computational complexity of the methodology and is beyond the aim of this study.

The set of conditional moments (9) can be expressed as an augmented set of unconditional moments if we assume that the conditioning information set can be reflected by the set of state variables z_t . Then, the set of unconditional moments is

$$E \left[\sum_{j=1}^K \beta^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \otimes z_t \right] = 0 \quad (11)$$

where \otimes denotes element by element multiplication. More specifically, expression (11) yields the following system of mn^2 conditions:

$$\phi_{h,s}^{\tilde{s}}(\mu, \omega) \equiv E \left[\sum_{j=1}^K \beta^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) z_{\tilde{s},t} \right] = 0, \quad (12)$$

where $\mu = (\lambda, \gamma, \eta)$ and $h = 1, \dots, m$, $s, \tilde{s} = 1, \dots, n$ and $z_{1,t} = 1$.

The main advantage of this approach is that the first order conditions of the maximization problem of a strategic investor with power utility yield a simple system of equations that is overidentified and provides a very intuitive empirical representation. This property is exploited in the econometric section to derive suitable estimators of the portfolio weights and carry out statistical tests of the specifications (3) and (6).

3 Econometric methods: estimation

This section presents suitable methods to estimate the optimal portfolio weights and the parameters driving the dynamics of the risk aversion coefficient. A suitable empirical representation of the Euler equation (12) is

$$\widehat{\phi}_{h,s}^{\tilde{s}}(\mu, \omega) \equiv \frac{1}{T-K} \sum_{t=1}^{T-K} e_{hs,t}(\mu, \omega) z_{\tilde{s},t} = 0 \quad (13)$$

with

$$e_{hs,t}(\mu, \omega) = \sum_{j=1}^K \beta^j \psi_{t,j}(z_s; \lambda_h, \gamma, \eta, \omega) \quad (14)$$

and T is the sample size used for estimating the model parameters.

For each $\omega \in [\omega_{\min}, \omega_{\max}]$, let $g(\mu, \omega)$ and $g_T(\mu, \omega)$ be the $mn^2 \times 1$ vectors that stack each of the sample moments $\phi_{h,s}^{\tilde{s}}(\mu, \omega)$ and $\hat{\phi}_{h,s}^{\tilde{s}}(\mu, \omega)$, respectively, indexed by h, s and \tilde{s} , with $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$. The idea behind GMM is to choose $\hat{\mu}_T$ so as to make the sample moments $g_T(\mu, \omega)$ as close to zero as possible. An important distinction with respect to the linear case is the existence of a threshold parameter ω that determines the presence of nonlinearities in the investor's strategic behavior. This parameter introduces a break in the individual's objective function that determines two regimes in the functional form of the risk aversion coefficient.

To estimate the model parameters in the general case given by absence of knowledge of the true population parameter ω_0 , we propose a two-step estimation procedure². First, for each ω , we define the set of parameter estimators $\hat{\mu}_T(\omega)$ of the true parameter vector $\mu \in \Theta$ as

$$\hat{\mu}_T(\omega) = \arg \min_{\mu \in \Theta} g'_T(\mu, \omega) V_T^{-1}(\omega) g_T(\mu, \omega) \quad (15)$$

where

$$\begin{aligned} V_T(\omega) &= \frac{1}{T-K} \sum_{t=1}^{T-K} e_{h_1 s_1, t}(\mu, \omega) e_{h_2 s_2, t}(\mu, \omega) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t} \\ &+ \frac{1}{T-K} \sum_{t=1}^{T-K} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K} e_{h_1 s_1, t}(\mu, \omega) e_{h_2 s_2, t'}(\mu, \omega) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t'} \end{aligned} \quad (16)$$

is a consistent estimator of $V_0(\omega) = E[g_T(\mu, \omega) g'_T(\mu, \omega)]$, a $mn^2 \times mn^2$, possibly random, non-

²A similar two-step procedure for estimation of the model parameters using GMM methods is proposed by Seo and Shin (2014).

negative definite weight matrix, whose rank is greater than or equal to mn . This estimator highlights the strong persistence in the covariance matrix $V_0(\omega)$. This persistence is due to the presence of serial correlation produced by considering a strategic investment horizon ($K > 1$) in the individual's objective function. The second step of the estimation process consists of finding the strategic horizon that minimizes the objective function on ω . More formally,

$$\hat{\omega}_T = \arg \min_{\omega \in [\omega_{\min}, \omega_{\max}]} \hat{\mu}_T(\omega). \quad (17)$$

The strategic horizon associated to the optimal $\hat{\omega}_T$ is given by $\hat{k}_T = \lceil \hat{\omega}_T K \rceil$. Applying standard results already derived in Chan (1993), Andrews (1993) and Hansen (2000) for OLS methods and in Seo and Shin (2014) for GMM, we state without formal proof that

$$\hat{\omega}_T \xrightarrow{p} \omega_0. \quad (18)$$

In this paper we are interested in making inference on the model parameters reflecting the dynamics of the optimal portfolio allocation (λ) and risk aversion coefficients (γ, η). For these parameters, a direct application of standard asymptotic theory for nonlinear threshold models and structural break detection models, see Hansen (1996, 2000), Gonzalo and Pitarakis (2002) and Gonzalo and Wolf (2005) for OLS procedures, and Seo and Shin (2014) for GMM estimation, we obtain the following result:

$$\sqrt{T} (\hat{\mu}_T(\hat{\omega}_T) - \mu) \xrightarrow{d} N \left(0, (D'(\omega) \Omega^{-1}(\omega) D(\omega))^{-1} \right) \quad (19)$$

with $\Omega(\omega) = E[g(\mu, \omega)g'(\mu, \omega)]$ and $D(\omega) \equiv D(\mu, \omega) = \frac{\partial g(\mu, \omega)}{\partial \mu}$ a function that is continuous in the vector μ .

4 Econometric methods: hypothesis testing

This section presents a threshold nonlinearity test to statistically assess whether there exist dynamics in the risk aversion coefficient that can be modeled as a two-regime piecewise linear process. Second, we exploit the overidentified system of equations (11) to propose a specification test for the parametric formulation of the risk aversion function (3) and the policy rule (6).

4.1 Threshold nonlinearity tests

Following the literature on threshold and structural break models we will distinguish two cases. One, in which the timing of the break ω_0 is known, and a second case, in which ω_0 is not identified under the null hypothesis. In both scenarios the null hypothesis corresponds to the case

$$H_0 : \eta_c = \eta_1 = \dots = \eta_{n-1} = 0 \text{ against } H_A : \eta_s \neq 0 \text{ for some } s = c, 1, \dots, n-1,$$

in the dynamic risk aversion coefficient (3). This composite test is standard for ω_0 known and appropriate test statistics can be deployed by exploiting the overidentified system of equations (11). More specifically, a suitable nonlinearity test for the null hypothesis is the likelihood ratio test

$$L_K(\omega_0) = (T - K) (s(\hat{\mu}_{0T}, \omega_0) - s(\hat{\mu}_T, \omega_0)) \quad (20)$$

with $s(\hat{\mu}_T, \omega_0) = g'_T(\hat{\mu}_T, \omega_0) \hat{V}_T^{-1}(\omega_0) g_T(\hat{\mu}_T, \omega_0)$. Similarly, $s(\hat{\mu}_{0T}, \omega_0)$ is the version of the statistic under the null hypothesis H_0 . It is important to note that the covariance matrix of the parameter estimators refers for both statistics $s(\hat{\mu}_T, \omega_0)$ and $s(\hat{\mu}_{0T}, \omega_0)$ to the same consistent estimator of the covariance matrix $V_0(\omega_0)$ estimated under the unrestricted model. A natural candidate robust to the presence of serial correlation in the sample moments is the sample

covariance matrix

$$\widehat{V}_T(\omega) = \frac{1}{T-K} \sum_{t=1}^{T-K} \widehat{e}_{h_1 s_1, t} \widehat{e}_{h_2 s_2, t} z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t} + \frac{1}{T-K} \sum_{t=1}^{T-K} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K} \widehat{e}_{h_1 s_1, t} \widehat{e}_{h_2 s_2, t'} z_{\widetilde{s}_1, t} z_{\widetilde{s}_2, t'} \quad (21)$$

with

$$\widehat{e}_{hs, t} = \sum_{j=1}^K \beta^j \psi_{t, j}(z_s; \widehat{\lambda}_h, \widehat{\gamma}_T, \widehat{\eta}_T, \omega), \quad (22)$$

where $\widehat{\gamma}(j) = \exp(\widehat{\gamma}'_T z_{t+j} + \widehat{\eta}'_T z_{t+j} 1(\omega_j > \omega))$ and

$$\psi_{t, j}(z_s; \widehat{\lambda}_h, \widehat{\gamma}_T, \widehat{\eta}_T, \omega) = \left(\sum_{i=1}^j \frac{z_{s, t+i-1} r_{h, t+i}^e}{1 + r_{t+i}^p(\widehat{\lambda}'_h z_{t+i-1})} \right) \left(\prod_{i=1}^j (1 + r_{t+i}^p(\widehat{\lambda}'_h z_{t+i-1})) \right)^{1-\widehat{\gamma}(j)}.$$

Under these conditions, it holds that

$$L_K(\omega) \xrightarrow{d} \chi_n^2 \quad (23)$$

with n the number of restrictions implied by the null hypothesis H_0 .

A similar testing procedure can be developed to assess the existence of linear dynamics in the risk aversion coefficient against constant risk aversion. To do this, we take as benchmark under the alternative hypothesis a simplified version of (3) given by $\gamma(j) = \exp(\gamma' z_{t+j})$. The relevant hypothesis is

$$H_0 : \gamma_1 = \dots = \gamma_{n-1} = 0 \text{ against } H_A : \gamma_s \neq 0 \text{ for some } s = 1, \dots, n-1, \quad (24)$$

with the vector $(\gamma_1, \dots, \gamma_{n-1})'$ denoting the parameters associated to the state variables $z_{1, t}, \dots, z_{n-1, t}$.

For the most interesting cases, such as testing for nonlinearity of the preferences of the long-term investor when ω_0 is not known, $\omega_0 \in [\omega_{\min}, \omega_{\max}]$ is a nuisance parameter that cannot

be identified under the null hypothesis. In this case Hansen (1996) shows that the composite nonlinearity test is nonstandard. As proposed by this author, see also Davies (1977, 1987) or Andrews and Ploberger (1994) in different contexts, hypothesis tests for nonlinearity can be based on different functionals of the relevant test statistic computed over the domain of the nuisance parameter. In our framework, the relevant test statistic is $l_K = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} L_K(\omega)$ with sup standing for the supremum functional. In this case, the statistic $s(\mu, \omega)$ is a function on $\omega \in [\omega_{\min}, \omega_{\max}]$. To formalize the asymptotic distribution of $L_K(\omega)$ we define the covariance function

$$\Sigma_0(\omega_1, \omega_2) = E [g_T(\mu, \omega_1) g_T'(\mu, \omega_2)] \quad (25)$$

and its empirical counterpart

$$\begin{aligned} \hat{\Sigma}_T(\omega_1, \omega_2) &= \frac{1}{T-K} \sum_{t=1}^{T-K} \hat{e}_{h_1 s_1, t}(\hat{\mu}_T, \omega_1) \hat{e}_{h_2 s_2, t}(\hat{\mu}_T, \omega_2) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t} \\ &+ \frac{1}{T-K} \sum_{t=1}^{T-K} \sum_{\substack{t'=1 \\ t' \neq t}}^{T-K} \hat{e}_{h_1 s_1, t}(\hat{\mu}_T, \omega_1) \hat{e}_{h_2 s_2, t'}(\hat{\mu}_T, \omega_2) z_{\tilde{s}_1, t} z_{\tilde{s}_2, t'} \end{aligned}$$

with $\omega_1, \omega_2 \in [\omega_{\min}, \omega_{\max}]$. These expressions are the functional counterparts of the covariance matrices $V_0(\omega)$ and $\hat{V}_T(\omega)$, respectively.

To derive the asymptotic distribution of the relevant test we define the processes $S_T(\hat{\mu}_T, \omega) = \sqrt{T-K} g_T(\hat{\mu}_T, \omega)$ and $S_{0T}(\hat{\mu}_{0T}, \omega) = \sqrt{T-K} g_T(\hat{\mu}_{0T}, \omega)$. Under some suitable regularity conditions on the uniform convergence of $\hat{\Sigma}_T(\omega_1, \omega_2)$ to $\Sigma_0(\omega_1, \omega_2)$ over its compact support, see Hansen (1996) for more technical details, the process $S_T(\hat{\mu}_T, \omega)$ converges weakly to a multivariate zero mean Gaussian process, $S(\mu, \omega)$, defined by the covariance function $\Sigma_0(\omega_1, \omega_2)$. Similarly, under the null hypothesis H_0 the process $S_{0T}(\hat{\mu}_{0T}, \omega)$ converges to a multivariate zero-mean Gaussian process $S_0(\mu_0, \omega)$. Therefore, under the null hypothesis, the process $L_K(\omega)$

converges weakly to the following chi-square process

$$L_0(\omega) = S'_0(\mu_0, \omega) \Sigma_0(\omega, \omega)^{-1} S_0(\mu_0, \omega) - S'(\mu, \omega) \Sigma_0(\omega, \omega)^{-1} S(\mu, \omega). \quad (26)$$

Consequently, the asymptotic distribution of the supremum functional is $l_0 = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} L_0(\omega)$. Since the null distribution of (26) depends upon the covariance function Σ_0 , critical values cannot be tabulated. To obtain the p -values of the test we derive a p -value transformation similar in spirit to the work of Hansen (1996), based on a multiplier bootstrap.

Let $F_0(\cdot)$ denote the distribution function of l_0 , and define $p_T = 1 - F_0(l_K)$. The above result shows that p_T converges in probability to $p_0 = 1 - F_0(l_0)$, that under the null hypothesis is uniform on $[0, 1]$. Thus the asymptotic null distribution of p_T is free of nuisance parameters. The rejection rule of our test is given by $p_T < \alpha$ with α the significance level and p_T the asymptotic p -value. The random variable l_0 can be written as a continuous functional of the Gaussian processes $S(\mu, \omega)$ and $S_0(\mu_0, \omega)$, which are completely described by the covariance function $\Sigma_0(\omega_1, \omega_2)$. To implement the p -value transformation, we operate conditional on the sample $\mathfrak{S} = \{(r'_{t+1}, z'_t)'\}_{t=1}^T$ and define the conditional multivariate mean-zero Gaussian processes \hat{S}_T and \hat{S}_{0T} . These processes can be generated by letting $\{v_t\}_{t=0}^{T-1}$ be *i.i.d.* $N(0, 1)$ random variables, and setting for $h = 1, \dots, m$ and $s, \tilde{s} = 1, \dots, n$, the following processes

$$\hat{S}_T(\hat{\mu}_T, \omega) = \frac{1}{\sqrt{T-K}} \sum_{t=1}^{T-K} \hat{e}_{hs,t} z_{\tilde{s},t} v_t. \quad (27)$$

Similarly, we have

$$\hat{S}_{0T}(\hat{\mu}_{0T}, \omega) = \frac{1}{\sqrt{T-K}} \sum_{t=1}^{T-K} \hat{e}_{hs,t}^0 z_{\tilde{s},t} v_t \quad (28)$$

with $\hat{e}_{hs,t}^0$ the version of $\hat{e}_{hs,t}$ in (22) obtained under the null hypothesis H_0 . The corresponding

conditional chi-square process is

$$\widehat{L}_K(\omega) = \widehat{S}'_{0T}(\widehat{\mu}_{0T}, \omega) \widehat{V}_T^{-1}(\omega) \widehat{S}_{0T}(\widehat{\mu}_{0T}, \omega) - \widehat{S}'_T(\widehat{\mu}_T, \omega) \widehat{V}_T^{-1}(\omega) \widehat{S}_T(\widehat{\mu}_T, \omega) \quad (29)$$

and the corresponding test statistic is $\widehat{l}_K = \sup_{\omega \in [\omega_{\min}, \omega_{\max}]} \widehat{L}_K(\omega)$. Finally, let \widehat{F}_0 denote the conditional distribution function of \widehat{l}_K and $\widehat{p}_T = 1 - \widehat{F}_0(\widehat{l}_K)$.

The introduction of the zero-mean random variable v_t implies that, conditionally, the covariance function of $\widehat{S}_T(\widehat{\mu}_T, \omega)$ is equal to $\widehat{\Sigma}_T(\omega, \omega)$, that is,

$$E \left[\frac{1}{T-K} \sum_{t=1}^{T-KT-K} \sum_{t'=1}^{T-KT-K} \widehat{e}_{hs,t} z_{\widetilde{s},t} \widehat{e}_{hs,t'} z_{\widetilde{s},t'} v_t v_{t'} \mid \mathfrak{S} \right] = \frac{1}{T-K} \sum_{t=1}^{T-KT-K} \sum_{t'=1}^{T-KT-K} \widehat{e}_{hs,t} z_{\widetilde{s},t} \widehat{e}_{hs,t'} z_{\widetilde{s},t'} E[v_t v_{t'} \mid \mathfrak{S}].$$

Following similar arguments to the proof of Theorem 2 in Hansen (1996), it can be shown that the quantity \widehat{p}_T is asymptotically equivalent to p_T under both the null and alternative hypotheses. The conditional distribution function \widehat{F}_T is not directly observable so neither is the random variable \widehat{p}_T . Nevertheless, these quantities can be approximated to any desired degree of accuracy using standard simulation techniques. The following algorithm shows the implementation of this p-value transformation. Let Ω_N define a grid of N points over the compact set $[\omega_{\min}, \omega_{\max}]$, and let ω_i for $i = 1, \dots, N$ be the set of equidistant points in such grid with $\omega_1 = \omega_{\min}$ and $\omega_N = \omega_{\max}$; for $j = 1, \dots, J$, execute the following steps:

- i) generate the sequence $\{v_{jt}\}_{t=1}^T$ *i.i.d.* random variables;
- ii) conditional on the sample $\mathfrak{S} = \{(r'_{t+1}, z'_t)'\}_{t=1}^T$, set the quantities $\widehat{S}_T^j(\widehat{\mu}_T, \omega_i)$ and $\widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i)$;
- iii) set $\widehat{L}_K^j(\omega_i) = \widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i) \widehat{V}_T^{-1}(\omega_i) \widehat{S}_{0T}^j(\widehat{\mu}_{0T}, \omega_i) - \widehat{S}_T^j(\widehat{\mu}_T, \omega_i) \widehat{V}_T^{-1}(\omega_i) \widehat{S}_T^j(\widehat{\mu}_T, \omega_i)$;
- iv) set $\widehat{l}_K^j = \sup_{\omega \in \Omega_N} \widehat{L}_K^j(\omega_i)$.

This gives a random sample $(\hat{l}_K^1, \dots, \hat{l}_K^J)$ from the conditional distribution \hat{F}_T . The percentage of these artificial observations which exceeds the actual test statistic l_K : $\hat{p}_T^J = \frac{1}{J} \sum_{j=1}^J 1(\hat{l}_K^j > l_K)$ is according to the Glivenko-Cantelli theorem a consistent approximation of \hat{p}_T as $J \rightarrow \infty$. In practice, the null hypothesis H_0 is rejected if $\hat{p}_T < \alpha$.

4.2 Specification tests

The system of equations defined in (12) entails the existence of testable restrictions of our econometric specification determined by the nonlinear risk aversion function (2) and the parametric portfolio weights (6). Estimation of $\mu = (\lambda, \gamma, \eta)$ sets to zero $mn + 2n$ linear combinations of the mn^2 sample orthogonality conditions $g_T(\mu, \omega)$ with $\omega \in [\omega_{min}, \omega_{max}]$. The correct specification of the model implies that, for a fixed ω_0 , there are $mn^2 - mn - 2n$ linearly independent combinations of $g_T(\hat{\mu}_T, \omega_0)$ that should be close to zero but are not exactly equal to zero. This hypothesis is tested using the Hansen test statistic (Hansen, 1982).

Let $s(\hat{\mu}_T, \omega_0) = g_T(\hat{\mu}_T, \omega_0)' \hat{V}_T^{-1}(\omega_0) g_T(\hat{\mu}_T, \omega_0)$, that under the null hypothesis of correct specification of the model, satisfies

$$s(\hat{\mu}_T, \omega_0) \xrightarrow{d} \chi_{mn^2 - mn - 2n}^2. \quad (30)$$

The null hypothesis of correct specification of the overidentified system of equations is rejected at a significance level α if the test statistic $s(\hat{\mu}_T, \omega_0)$ is greater than the critical value $\chi_{mn^2 - mn - 2n, 1 - \alpha}^2$. In practice, the parameter ω_0 can be replaced by the estimator $\hat{\omega}_T$ obtained from (17). A similar specification test can be developed to test the linear version of the above model against the model exhibiting constant risk aversion. In this case the relevant asymptotic condition is

$$s(\hat{\mu}_{0T}) \xrightarrow{d} \chi_{mn^2 - mn - n}^2, \quad (31)$$

with $s(\hat{\mu}_{0T}) = g_T(\hat{\mu}_{0T})' \hat{V}_{0T}^{-1} g_T(\hat{\mu}_{0T})$, where $g_T(\hat{\mu}_{0T})$ and \hat{V}_{0T} are the versions of the sample moment conditions and the empirical covariance function (21) obtained from the estimation of the linear dynamic model.

5 Empirical application

In this section we analyze the optimal portfolio decisions and risk aversion dynamics of an strategic individual with objective function characterized by the time preference parameter $\beta = 0.95$ and investment horizons between two ($K = 24$) and four years ($K = 48$)³. Our aim is to compare the optimal portfolio choices across investment horizons of three different types of strategic investors: individuals that exhibit a constant relative risk aversion coefficient, individuals that exhibit a relative risk aversion coefficient that varies linearly over time according to the dynamics of our set of state variables, and finally, individuals that exhibit different degrees of risk aversion to the short and the long term via a threshold specification on the investment horizon.

We consider a tactical asset allocation setting characterized by a portfolio of stocks, bonds and the one-month real Treasury bill rate. As in Campbell et al. (2003), we do not impose short-selling restrictions. Our data covers the period January 1980 to December 2010. Monthly data are collected from Bloomberg on the S&P 500 and G0Q0 Bond Index. The G0Q0 Bond Index is a Bank of America and Merrill Lynch U.S. Treasury Index that tracks the performance of U.S. dollar denominated sovereign debt publicly issued by the U.S. government in its domestic market. The nominal yield on the U.S. one-month risk-free rate is obtained from the Fama and French database, and the consumer price index (CPI) time series and the yield of the Moody's Baa- and Aaa-rated corporate bonds from the U.S. Federal Reserve.

³Other investment horizons such as $K = 12$ and $K = 60$ are available upon request.

The time-variation of the investment opportunity set is described by a set of state variables that have been identified in the empirical literature as potential predictors of the excess stock and bond returns and the short-term ex-post real interest rates. These variables are the detrended short-term interest rate (Campbell, 1991), the U.S. credit spread (Fama and French, 1989), the S&P 500 trend (Aït-Sahalia and Brandt, 2001) and the one-month average of the excess stock and bond returns (Campbell et al., 2003). The detrended short-term interest rate detrends the short-term rate by subtracting a 12-month backwards moving average. The U.S. credit spread is defined as the yield difference between Moody’s Baa- and Aaa-rated corporate bonds. The S&P 500 trend, or momentum, state variable is defined as the difference between the log of the current S&P 500 index level and the average index level over the previous 12 months. We demean and standardize all the state variables in the optimization process (Brandt et al, 2009).

[Insert Table 1 about here]

Table 1 reports the sample statistics of the annualized excess stock return, excess bond return and short-term ex-post real interest rates. The bond market outperforms the stock market during this period. In particular, the excess return on the bond index is higher than for the S&P 500 and exhibits a lower volatility entailing a Sharpe ratio almost three times higher for bonds than stocks. Additionally, the excess bond return has larger skewness and lower kurtosis. This anomalous outperformance of the G0Q0 index versus the S&P 500 is mainly explained by the last part of the sample and the consequences of the subprime crisis on the valuation of the different risky assets.

5.1 Empirical results

The parameter estimates driving the optimal portfolio rules and dynamic risk aversion coefficients are estimated using a two-step Gauss-Newton type algorithm with numerical derivatives. The method is implemented in Matlab and code is available upon request. In a first stage we initialize the covariance matrix \widehat{V}_T with the matrix $I_{mn} \otimes Z'Z$, and in a second stage, after obtaining a first set of parameter estimates, we repeat the estimation replacing this matrix by a trimmed version of (16). In particular, we use a Newey-West estimator of the matrices $V_0(\omega)$ with $K = 12$ lags for different choices of ω within the compact set. The covariance matrix $\widehat{V}_T(\omega)$ is also used to perform the different threshold nonlinearity and specification tests described below. Tables 2 to 4 report estimates of the model parameters (optimal portfolio weights and risk aversion coefficients) for the three different types of investors. The first column contains the estimates of the nonlinear process distinguishing between the short and the long term. The second column reports the parameter estimates of a simplified version of this model characterized by linear dynamics in the risk aversion coefficient. The third column contains the benchmark static model employed in the literature.

The empirical analysis presented below reveals four main features that are common across investment horizons. First, the period that separates the short from the long term is found to be between the seventh and eighth month of the investment horizon. Second, the different likelihood ratio tests developed above provide strong statistical evidence of the presence of dynamics and nonlinearities in the individuals' risk aversion coefficient. In particular, we reject the null hypothesis of a constant risk aversion coefficient when compared against a linear dynamic risk aversion coefficient. Similarly, our novel nonlinearity likelihood ratio test (20) also provides substantial evidence of the presence of nonlinearities in risk aversion when compared to the linear dynamic case. These differences are more relevant as the investment horizon increases. Third, we observe that the allocation to bonds and stocks is negatively correlated. This finding

is indicative of the existence of flight to quality effects from stocks to bonds especially during market distress episodes. Fourth, during these periods, we observe a significant increase in the allocation to bonds from the constant risk aversion model to the nonlinear dynamic model. In contrast, the optimal allocation to stocks is robust to the form of the risk aversion coefficient even under market distress. This result can be rationalized by the role of liquidity in the allocation to the S&P 500 index.

More specifically, Tables 2 to 4 show ample evidence that the optimal portfolio weights are driven by the dynamics of the state variables. For the two-year investment horizon ($K = 24$), we observe that increases of the detrended short-term interest rate and the one-month average of the excess stock and bond returns have a positive effect on the allocation to the S&P 500 index. The U.S. credit spread also has a positive effect on the optimal allocation to stocks but is not statistically significant. The S&P 500 trend has a negative effect on the magnitude of the allocation to stocks but is not statistically significant. In contrast, all of the state variables have a negative and statistically significant effect on the G0B0 bond index. This observation implies that increases in the value of the state variables entail short positions on the bond index and a negative correlation between the allocation to stocks and bonds. Positive allocations to stocks are corresponded by negative allocations to bonds, with both allocations determined by the evolution of the state variables. For larger investment horizons, the results are consistent with the two-year investment horizon. Interestingly, for $K = 36$ and $K = 48$, we observe an increase in the magnitude of the λ parameter estimates suggesting larger exposures to both stocks and bonds as the investment horizon increases.

Tables 2 to 4 also reveal interesting insights about risk aversion. The constant risk aversion coefficient increases from 47 to 60 as the investment horizon increases. The role of the state variables in driving risk aversion also becomes more significant as K increases. Thus, for the two-year investment horizon, the U.S. credit spread and the S&P 500 trend are statistically

significant determinants of risk aversion. As the investment horizon expands, the rest of state variables gain importance in driving the risk aversion coefficient.

[Insert Tables 2 to 4 about here]

The two-step estimation procedure establishes the presence of a structural break in the function $\gamma(j)$ at the seventh lag ($\widehat{k}_T = 7$) of the individual's long-term horizon. This result is rather robust across specifications of the investment horizon and provides a clear distinction between the short and the long term with regards to the risk aversion specification. The existence of two regimes in the risk aversion function is further supported by the results of the different likelihood ratio tests developed in Section 4 comparing the linear model against the piecewise linear model, and the model with constant risk aversion against the linear dynamic model. In particular, for $K = 24$, the value of the test statistic (20) is 11.93 that yields a p-value of 0.018. The test statistic for $H_0 : \gamma_1 = \dots = \gamma_{n-1} = 0$ is very high and the corresponding p-value is zero. For larger investment horizons we find stronger statistical evidence to reject the null hypotheses of constant and linear risk aversion, respectively, with p-values of the likelihood ratio tests equal to zero across hypothesis tests.

To illustrate the behavior of the risk aversion coefficient we report in Figures 1 to 3 the dynamics of the constant, linear and piecewise linear risk aversion functions. The top panel reports the constant and linear dynamic risk aversion coefficient (2) defined as $\gamma(j) = \exp(\widehat{\gamma}_c)$ and $\gamma(j) = \exp(\widehat{\gamma}'z_{t+j})$, respectively. The bottom panel plots the nonlinear version of the risk aversion function. For comparison purposes, we report separately the short-term $\gamma(j) = \exp(\widehat{\gamma}'z_{t+j})$ and the long-term $\gamma(j) = \exp(\widehat{\gamma}'z_{t+j} + \widehat{\eta}'z_{t+j}1(j > \widehat{k}_T))$.

[Insert Figures 1 to 3 about here]

The top panels of Figures 1 to 3 report notable fluctuations in risk aversion during the first half of the 1980 decade. The charts also reveal a larger degree of risk aversion for the

dynamic model than for the constant risk aversion model. This period corresponds to highly inflationary episodes produced by worldwide political instabilities and a sharp increase in oil prices that led to a worldwide economic recession. This trend is compensated during the period 2000 – 2006 that corresponds to the Great Moderation. This period was characterised by economic stability, strong growth, low inflation and low and stable interest rates. During this episode the dynamic risk aversion coefficient is below the constant risk aversion coefficient γ_c . The comparison of the top panels across Figures 1 to 3 also reveals increases in the magnitude of the dynamics of risk aversion across investment horizons. The bottom panels are also very illustrative of the additional effect of the long-term segment to the risk aversion function. In these graphs, we observe spikes in long-term risk aversion compared to short-term risk aversion from the observation 180 to 230. The latter period corresponds to the second half of the 1990 decade and the start of the new millennium. There is another spike in long-term risk aversion from 2005 onwards. These figures also reflect important differences between short-term and long-term risk aversion. Following conventional wisdom, individuals are more risk-averse to the long term than to the short term. Short-term risk aversion ranges between values of 35 to 65 whereas long-term risk aversion can take much larger values. The only exception being the period comprising the first years of the 1980 decade. In that period, our empirical exercise reveals large and similar values of risk aversion between the short and the long term.

To provide further empirical evidence on the role of risk aversion in determining optimal portfolio policies for long-term investors we report in Figures 4 to 6 the dynamics of the optimal portfolio allocations to stocks (α_{st}) and bonds (α_{bt}) for the three investment horizons over the period 1980 to 2010. Figures 7 to 9 complete the analysis by focusing on the recent 2007-2010 financial crisis episode. The top (bottom) panels report the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy, the dotted red line to the dynamic linear strategy and the solid blue line to

the constant risk aversion strategy.

[Insert Figures 4 to 9 about here]

The results suggest that distinguishing between these three scenarios does not play a fundamental role in determining the magnitude of the allocation to stocks. During crisis episodes we observe a significant drop in the optimal allocation to stocks, however, this result is rather uniform across risk aversion scenarios. This result can be rationalized by the liquidity exhibited by the S&P 500 index that prevents large sudden fluctuations in asset prices. Risk aversion plays a more important role in determining the optimal allocation to bonds, thus, we can appreciate significant increases in the optimal share of bonds in the investment portfolio when comparing the constant and nonlinear risk aversion investment strategies. More specifically, the nonlinear dynamic case allocates higher proportions of wealth to bonds during market distress episodes such as the first decade of 1980 and the 2007 – 2008 financial crisis suggesting that the increased uncertainty towards the long term might have been responsible for the flight to quality from stocks to bonds observed during these periods.

6 Conclusion

This paper studies the long-term asset allocation problem of an individual with risk aversion coefficient that *i*) varies with economic conditions, and *ii*) exhibits different risk attitudes towards the short and the long term. To do this, we propose a parametric linear portfolio policy that accommodates an arbitrarily large number of assets in the portfolio and a piecewise linear risk aversion coefficient. In this framework, individuals' risk aversion is driven by macroeconomic and financial conditions and exhibits nonlinearities produced by different views on the short and long term.

The empirical application to a tactical portfolio of three assets - a one-month Treasury bill as riskless security, a long-term bond, and an equity portfolio finds overwhelming empirical evidence of the presence of dynamics in the risk aversion coefficient. We provide statistical evidence of the existence of two regimes in the risk aversion function. The effect of long-term risk aversion is more relevant in determining the optimal allocation to bonds and more statistically significant as the investment horizon increases. The analysis of the optimal portfolio weights highlights the role of the state variables. More specifically, we find that the detrended short term interest rate and the one-month average of the excess stock and bond returns have a positive effect on the allocation to stocks and a negative effect on the allocation to bonds. The latter allocation is also influenced by the credit spread on U.S. bonds and the S&P 500 trend. These findings provide empirical evidence of a strong negative correlation between the allocation to stocks and bonds that can be interpreted during crisis episodes as evidence of flight to quality from stock markets to debt markets.

Further research is going in two directions. We consider other forms of nonlinearities in the risk aversion specification, and entertain a larger set of state variables via a factor augmentation model.

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Tables and figures

asset	Mean	Volatility	Sharpe ratio	Skewness	Kurtosis
S&P500 index	0.027	0.131	0.210	-1.120	4.88
G0Q0 Bond index	0.029	0.056	0.510	0.151	2.170
r_f	0.018	0.021	—	0.380	3.16

Table 1. Summary statistics of the excess stock return, excess bond return and short-term ex-post real interest rates over the period January 1980 to December 2010. The return horizon is one month. Mean and volatility are expressed in annualized terms.

stock parameters				bond parameters			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
$\lambda_{s,c}$	0.081 [0.013]	0.085 [0.000]	0.116 [0.000]	$\lambda_{b,c}$	0.206 [0.005]	0.164 [0.005]	0.186 [0.000]
$\lambda_{s,1}$	0.049 [0.092]	0.049 [0.010]	0.036 [0.035]	$\lambda_{b,1}$	-0.081 [0.108]	-0.096 [0.000]	-0.113 [0.000]
$\lambda_{s,2}$	-0.009 [0.705]	-0.019 [0.279]	-0.014 [0.307]	$\lambda_{b,2}$	-0.384 [0.000]	-0.392 [0.000]	-0.396 [0.000]
$\lambda_{s,3}$	0.033 [0.168]	0.018 [0.208]	0.017 [0.135]	$\lambda_{b,3}$	-0.319 [0.001]	-0.298 [0.000]	-0.291 [0.000]
$\lambda_{s,4}$	0.291 [0.000]	0.300 [0.000]	0.346 [0.000]	$\lambda_{b,4}$	-0.124 [0.051]	-0.114 [0.004]	-0.096 [0.009]
short term regime				long term regime			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
γ_c	3.822 [0.000]	3.984 [0.000]	3.849 [0.000]	η_c	0.557 [0.197]	—	—
γ_1	-0.068 [0.490]	-0.074 [0.245]	—	η_1	-0.153 [0.431]	—	—
γ_2	0.289 [0.052]	0.150 [0.002]	—	η_2	-0.118 [0.670]	—	—
γ_3	0.267 [0.013]	0.241 [0.000]	—	η_3	0.287 [0.327]	—	—
γ_4	0.195 [0.401]	0.103 [0.418]	—	η_4	-0.138 [0.657]	—	—
\hat{k}_T	7						
p-value	[0.018]	[0.000]					

Table 2. $K = 24$ and $\beta = 0.95$. The first column contains the estimates of the nonlinear process (2) that separates the short and long term. The second column reports the parameter estimates of the model with $\gamma(j)$ determined by a linear process. The third column contains the case corresponding to constant risk aversion. The parameters $\lambda_{s,i}$ with $i = c, 1, 2, 3, 4$ reflect the sensitivity of the state variables z_{it} to the optimal allocation to the S&P 500 index. The parameters $\lambda_{b,i}$ reflect the sensitivity of the state variables z_t to the optimal allocation to the G0Q0 Bond Index. The state variables defining z_{it} are a constant, the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns. P-values are in squared brackets.

stock parameters				bond parameters			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
$\lambda_{s,c}$	0.097 [0.001]	0.108 [0.000]	0.125 [0.000]	$\lambda_{b,c}$	0.214 [0.002]	0.185 [0.000]	0.202 [0.000]
$\lambda_{s,1}$	0.063 [0.037]	0.066 [0.000]	0.046 [0.015]	$\lambda_{b,1}$	-0.101 [0.018]	-0.111 [0.000]	-0.117 [0.000]
$\lambda_{s,2}$	-0.012 [0.681]	-0.015 [0.326]	-0.009 [0.472]	$\lambda_{b,2}$	-0.375 [0.000]	-0.408 [0.000]	-0.411 [0.000]
$\lambda_{s,3}$	0.017 [0.497]	-0.003 [0.774]	-0.003 [0.782]	$\lambda_{b,3}$	-0.373 [0.000]	-0.320 [0.000]	-0.306 [0.000]
$\lambda_{s,4}$	0.298 [0.000]	0.322 [0.000]	0.358 [0.000]	$\lambda_{b,4}$	-0.126 [0.047]	-0.122 [0.000]	-0.090 [0.005]
short term regime				long term regime			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
γ_c	3.799 [0.000]	3.959 [0.000]	3.861 [0.000]	η_c	0.645 [0.049]	—	—
γ_1	-0.119 [0.222]	-0.106 [0.020]	—	η_1	-0.124 [0.507]	—	—
γ_2	0.312 [0.061]	0.152 [0.001]	—	η_2	-0.125 [0.704]	—	—
γ_3	0.289 [0.004]	0.275 [0.000]	—	η_3	0.397 [0.141]	—	—
γ_4	0.126 [0.589]	-0.030 [0.801]	—	η_4	-0.247 [0.518]	—	—
\hat{k}_T	7						
p-value	[0.000]	[0.000]					

Table 3. $K = 36$ and $\beta = 0.95$. The first column contains the estimates of the nonlinear process (2) that separates the short and long term. The second column reports the parameter estimates of the model with $\gamma(j)$ determined by a linear process. The third column contains the case corresponding to constant risk aversion. The parameters $\lambda_{s,i}$ with $i = c, 1, 2, 3, 4$ reflect the sensitivity of the state variables z_{it} to the optimal allocation to the S&P 500 index. The parameters $\lambda_{b,i}$ reflect the sensitivity of the state variables z_t to the optimal allocation to the G0Q0 Bond Index. The state variables defining z_{it} are a constant, the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns. P-values are in squared brackets.

stock parameters				bond parameters			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
$\lambda_{s,c}$	0.118 [0.000]	0.125 [0.000]	0.134 [0.000]	$\lambda_{b,c}$	0.138 [0.005]	0.112 [0.000]	0.113 [0.000]
$\lambda_{s,1}$	0.034 [0.080]	0.039 [0.000]	0.023 [0.018]	$\lambda_{b,1}$	-0.058 [0.045]	-0.057 [0.000]	-0.066 [0.000]
$\lambda_{s,2}$	-0.038 [0.064]	-0.039 [0.000]	-0.033 [0.000]	$\lambda_{b,2}$	-0.268 [0.000]	-0.295 [0.000]	-0.296 [0.000]
$\lambda_{s,3}$	0.021 [0.275]	0.013 [0.084]	0.014 [0.046]	$\lambda_{b,3}$	-0.300 [0.000]	-0.253 [0.000]	-0.284 [0.000]
$\lambda_{s,4}$	0.257 [0.000]	0.265 [0.000]	0.283 [0.000]	$\lambda_{b,4}$	-0.055 [0.253]	-0.043 [0.063]	-0.031 [0.158]
short term regime				long term regime			
θ	nonlinear	linear	constant	θ	nonlinear	linear	constant
γ_c	4.007 [0.000]	4.190 [0.000]	4.100 [0.000]	η_c	0.605 [0.081]	—	—
γ_1	-0.061 [0.553]	-0.073 [0.056]	—	η_1	-0.180 [0.153]	—	—
γ_2	0.338 [0.024]	0.212 [0.000]	—	η_2	-0.223 [0.493]	—	—
γ_3	0.197 [0.035]	0.156 [0.000]	—	η_3	0.374 [0.121]	—	—
γ_4	-0.014 [0.952]	-0.029 [0.777]	—	η_4	-0.064 [0.851]	—	—
\hat{k}_T	7						
p-value	[0.000]	[0.000]					

Table 4. $K = 48$ and $\beta = 0.95$. The first column contains the estimates of the nonlinear process (2) that separates the short and long term. The second column reports the parameter estimates of the model with $\gamma(j)$ determined by a linear process. The third column contains the case corresponding to constant risk aversion. The parameters $\lambda_{s,i}$ with $i = c, 1, 2, 3, 4$ reflect the sensitivity of the state variables z_{it} to the optimal allocation to the S&P 500 index. The parameters $\lambda_{b,i}$ reflect the sensitivity of the state variables z_t to the optimal allocation to the G0Q0 Bond Index. The state variables defining z_{it} are a constant, the detrended short-term interest rate, the U.S. credit spread, the S&P 500 trend and the one-month average of the excess stock and bond returns. P-values are in squared brackets.

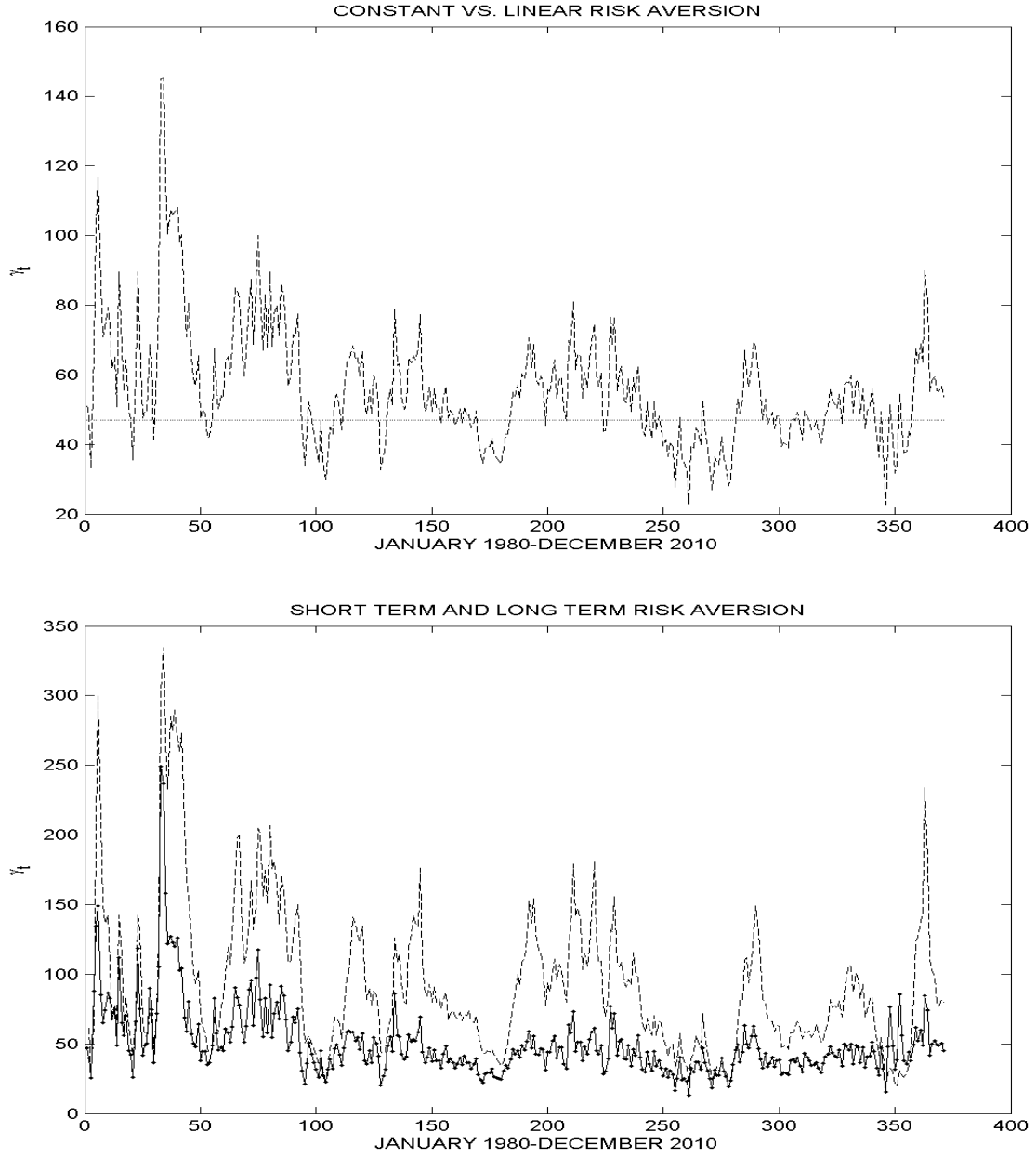


Figure 1. $K = 24$ investment periods. Top panel: The flat line for the model with constant risk aversion and the dashed line for the model with linear risk aversion dynamics. Bottom panel: The dotted line for short term dynamics of risk aversion and the dashed line for long-term dynamics.

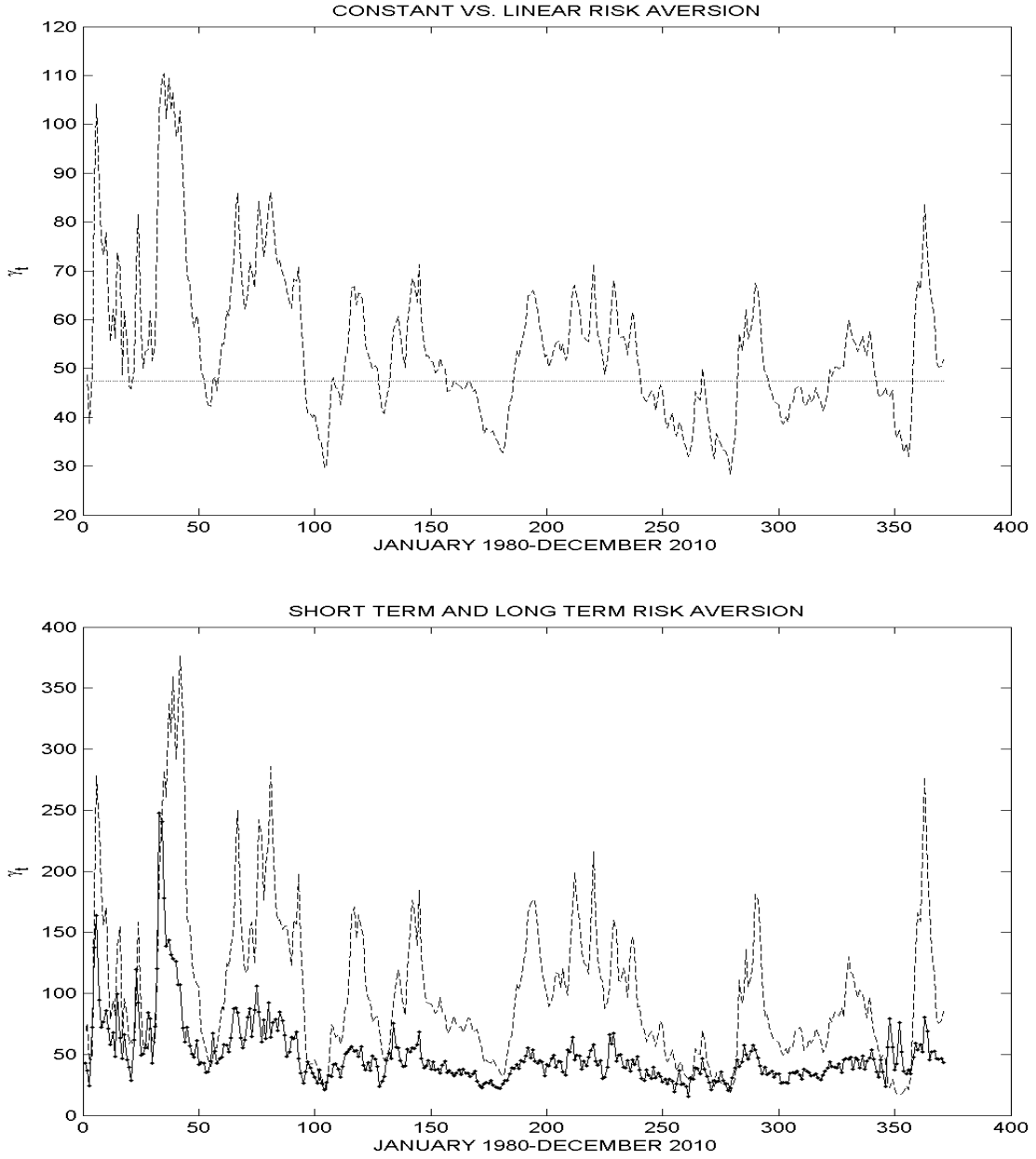


Figure 2. $K = 36$ investment periods. Top panel: The flat line for the model with constant risk aversion and the dashed line for the model with linear risk aversion dynamics. Bottom panel: The dotted line for short term dynamics of risk aversion and the dashed line for long-term dynamics.

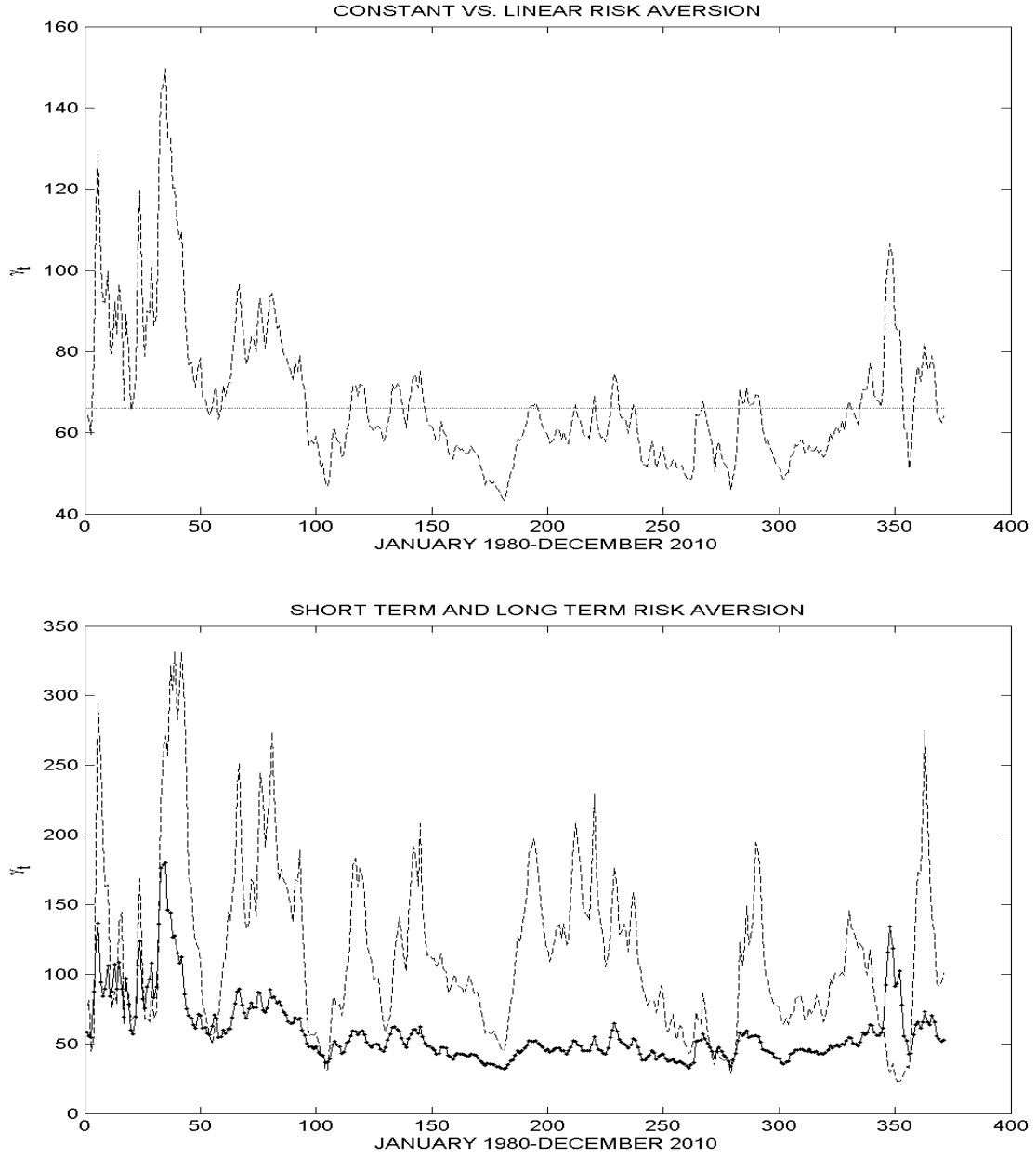


Figure 3. $K = 48$ investment periods. Top panel: The flat line for the model with constant risk aversion and the dashed line for the model with linear risk aversion dynamics. Bottom panel: The dotted line for short term dynamics of risk aversion and the dashed line for long-term dynamics.

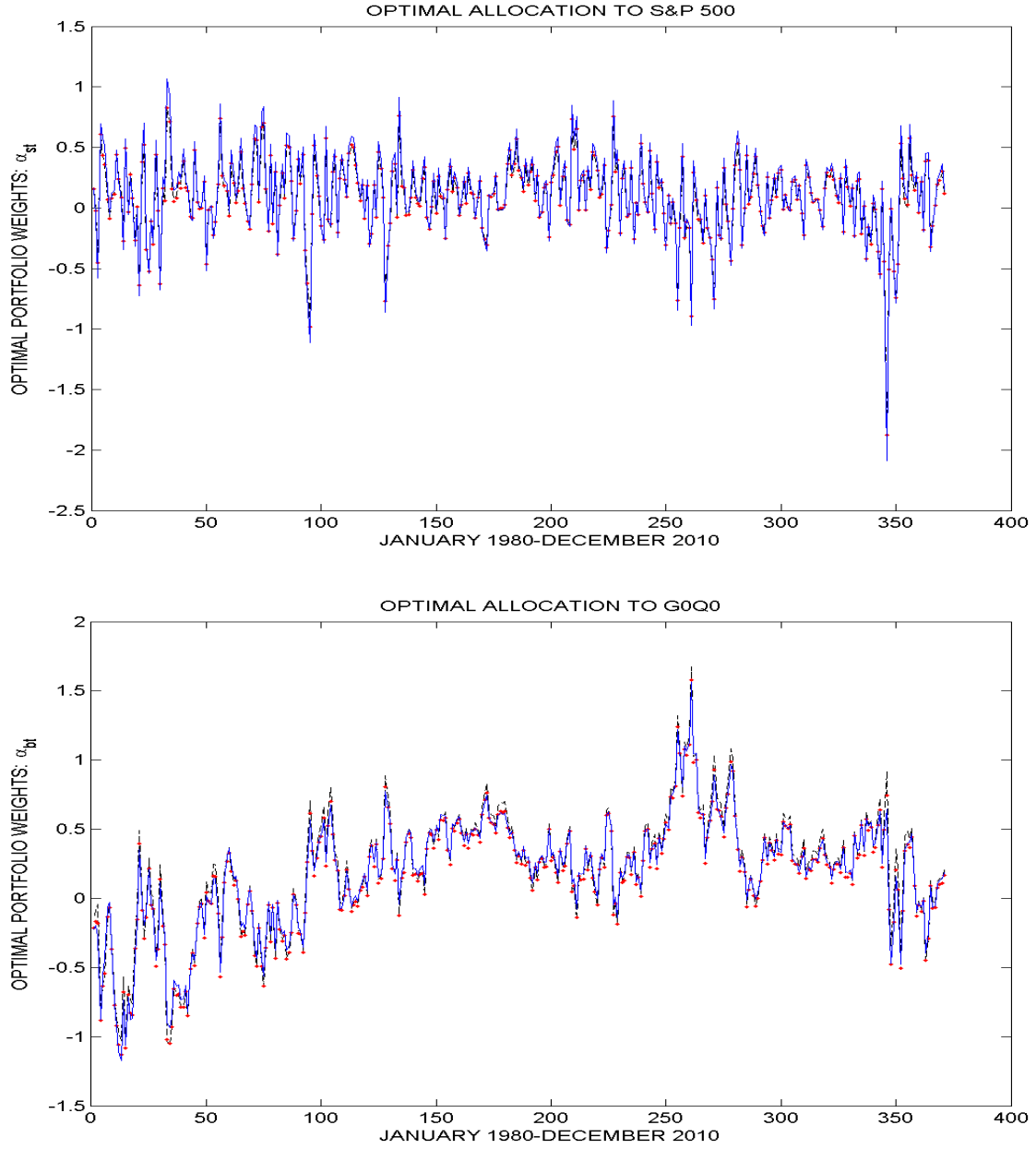


Figure 4. $K = 24$ investment periods over the period 1980-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy, the dotted red line to the dynamic linear strategy and the solid blue line to the constant risk aversion strategy.

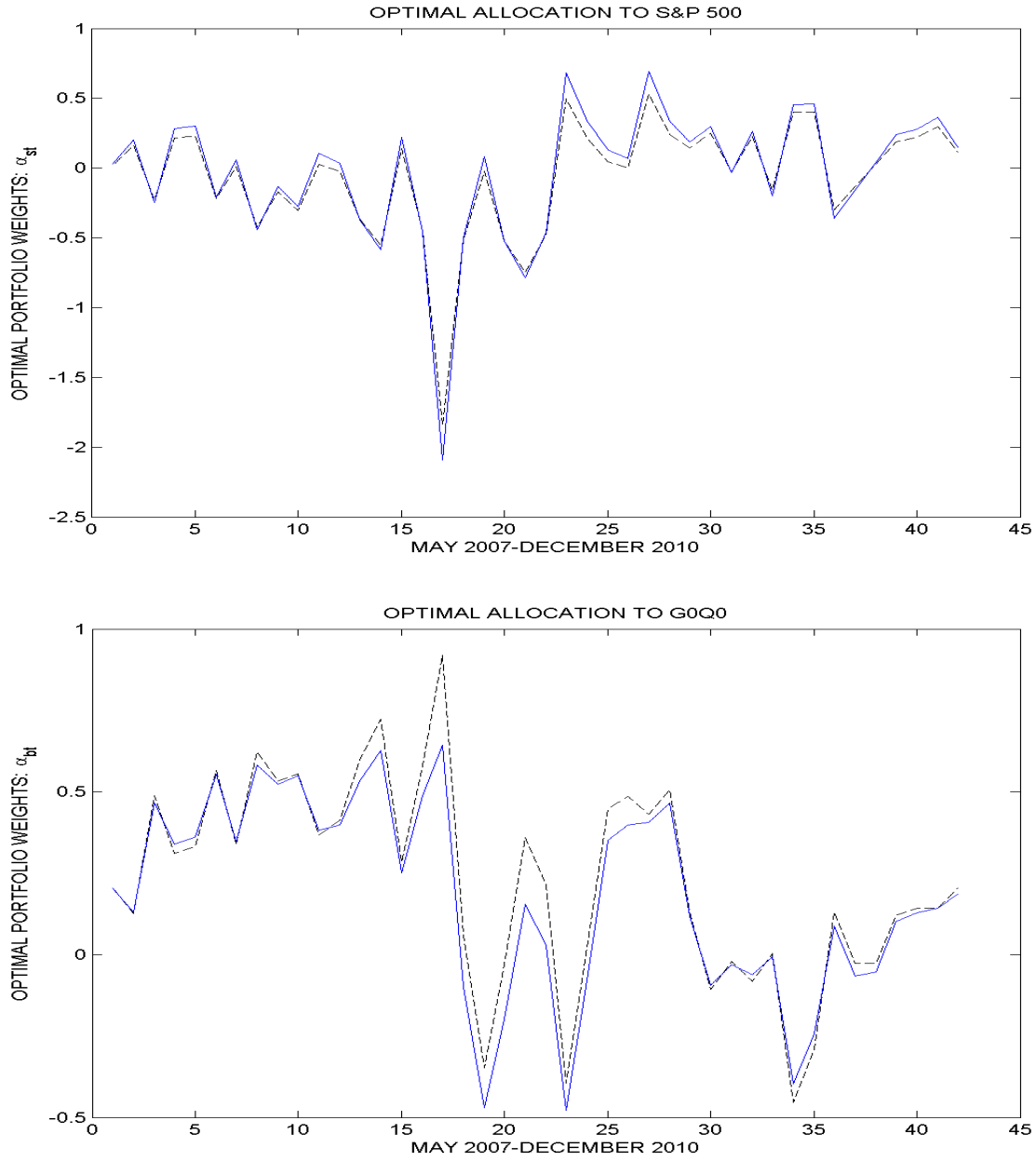


Figure 5. $K = 24$ investment periods over the period 2007-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy and the solid blue line to the constant risk aversion strategy.

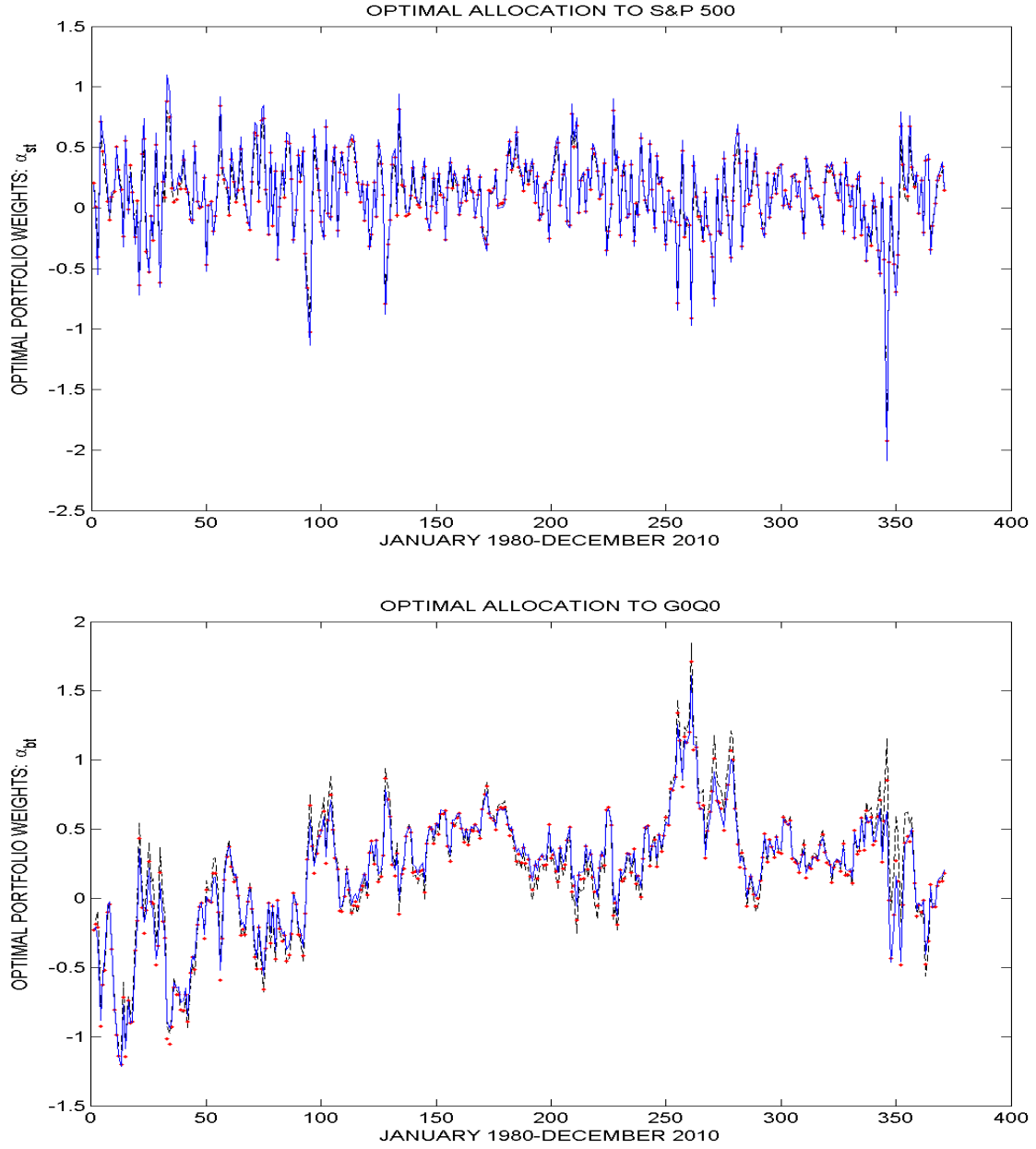


Figure 6. $K = 36$ investment periods over the period 1980-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy, the dotted red line to the dynamic linear strategy and the solid blue line to the constant risk aversion strategy.

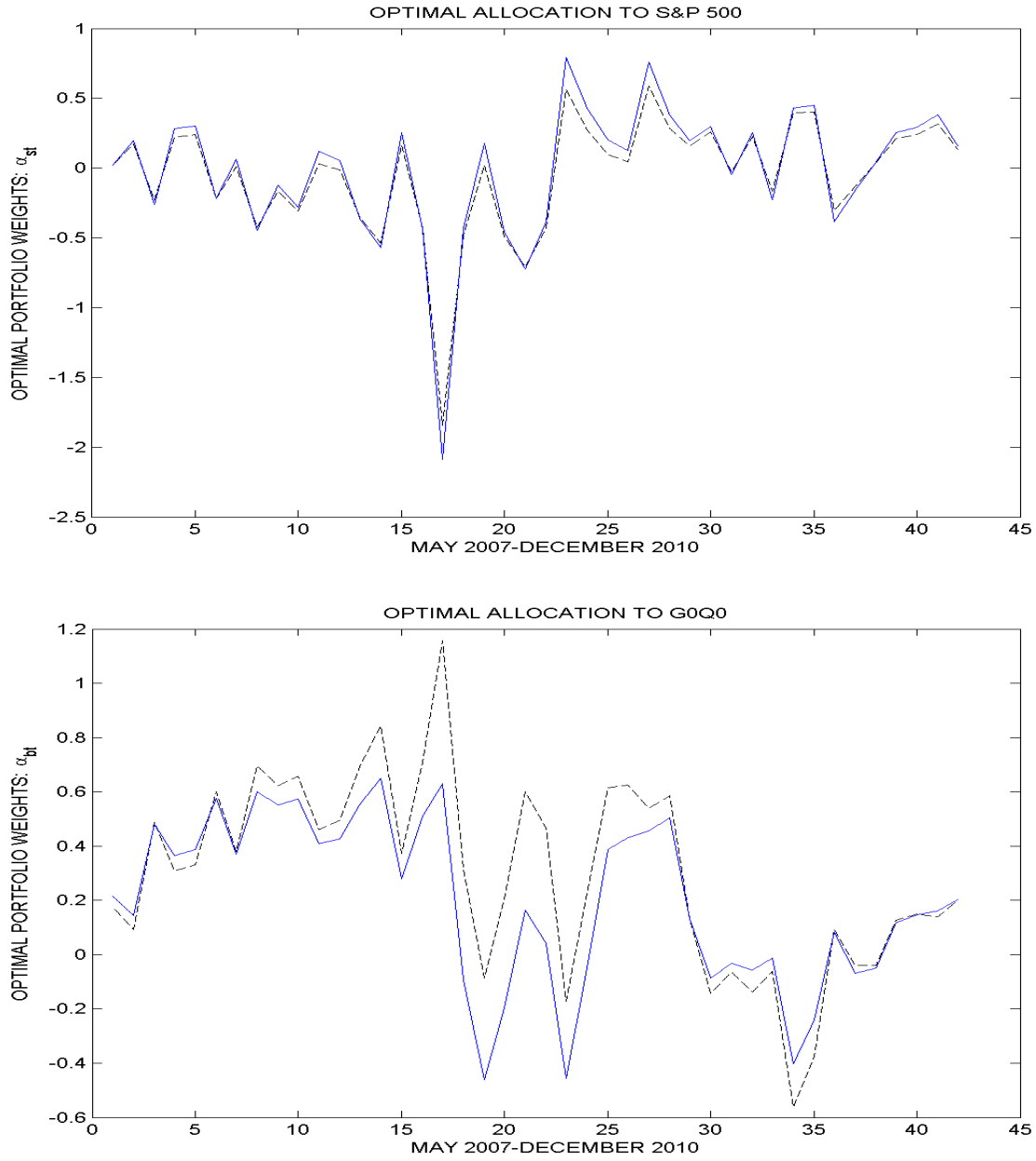


Figure 7. $K = 36$ investment periods over the period 2007-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy and the solid blue line to the constant risk aversion strategy.

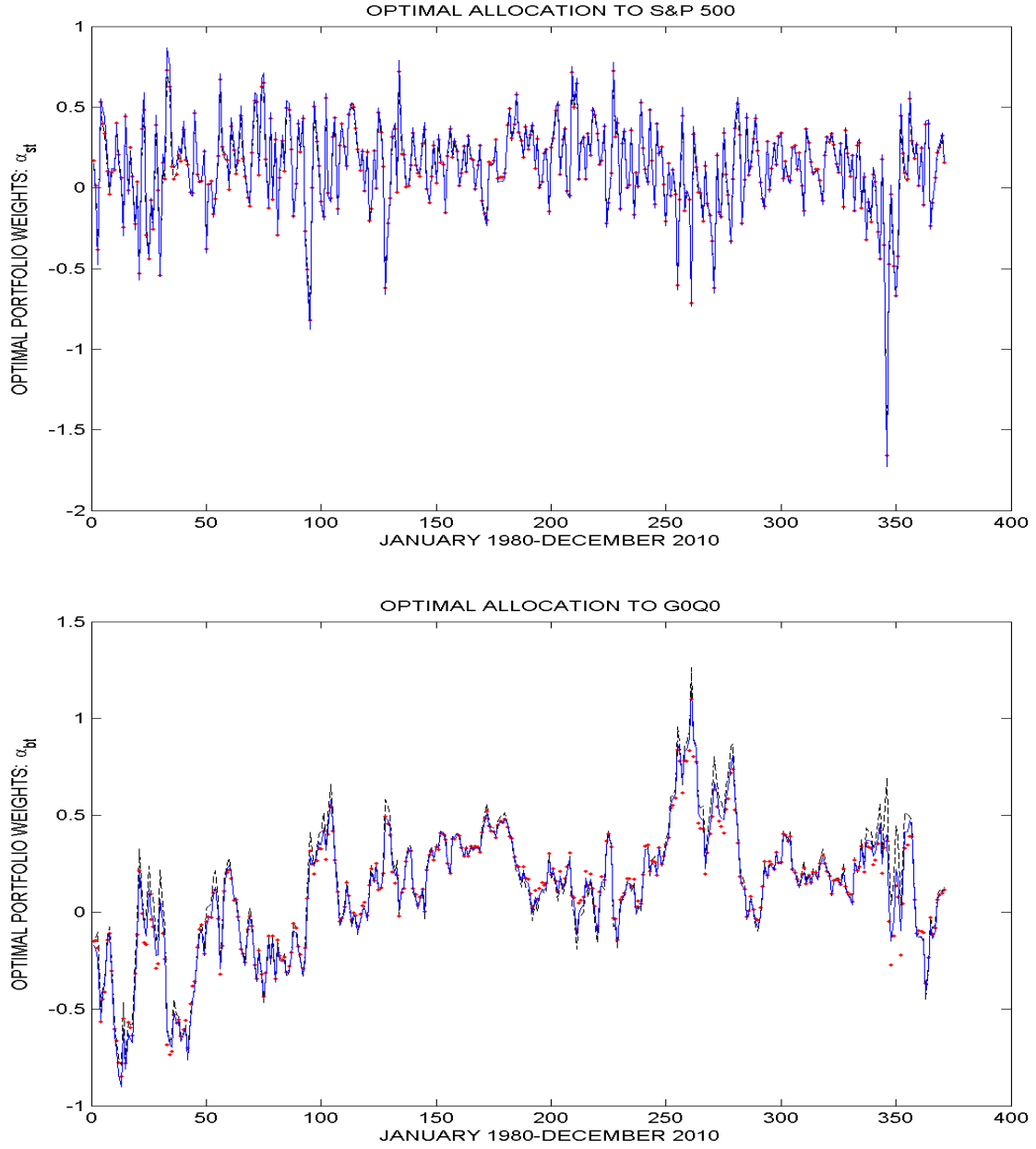


Figure 8. $K = 48$ investment periods over the period 1980-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy, the dotted red line to the dynamic linear strategy and the solid blue line to the constant risk aversion strategy.

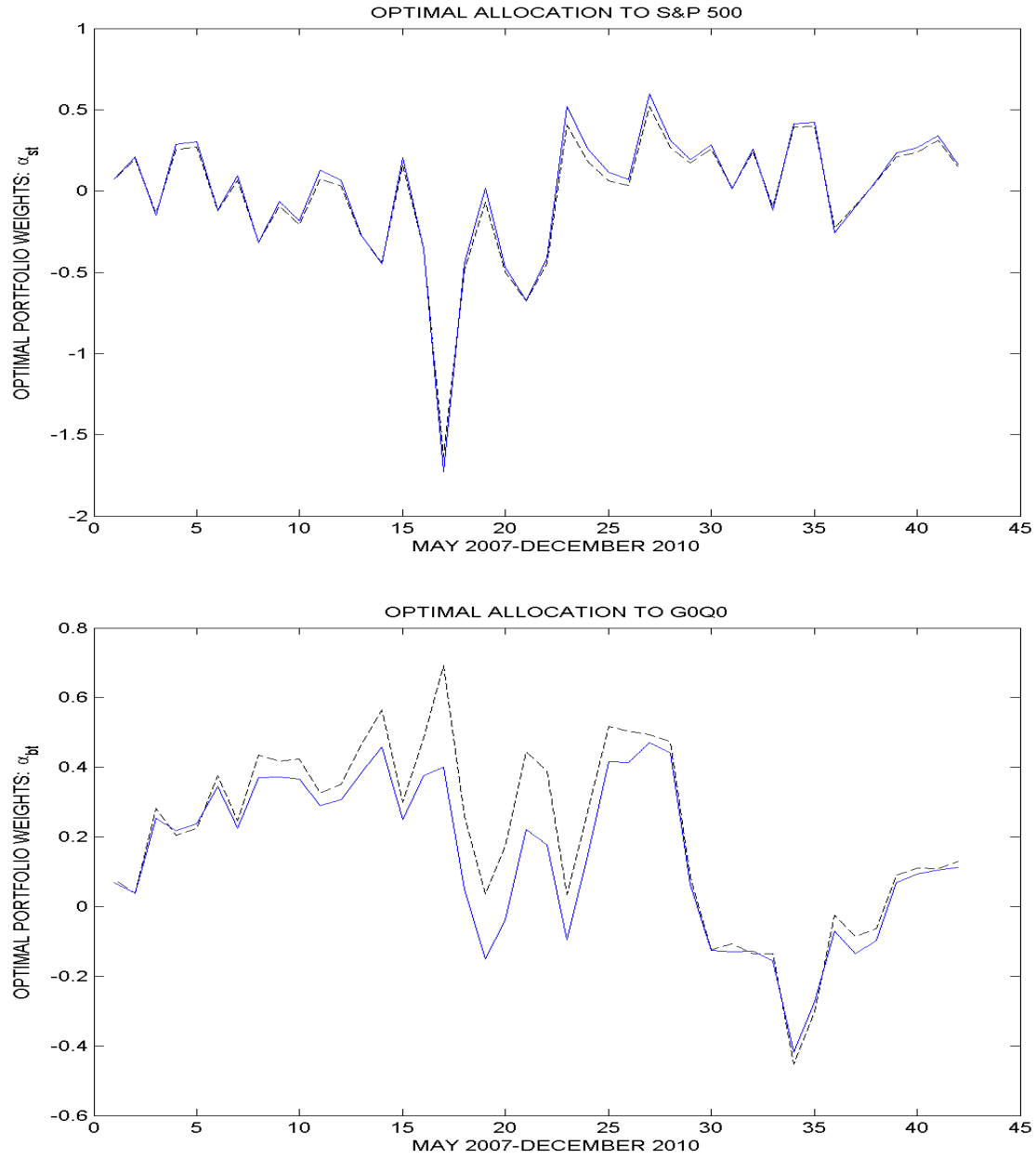


Figure 9. $K = 48$ investment periods over the period 2007-2010. The top (bottom) panel reports the optimal allocation to stocks (bonds) for each of the investment strategies. The dashed black line corresponds to the dynamic nonlinear strategy and the solid blue line to the constant risk aversion strategy.